

HEPTAGONAL KNOTS AND RADON PARTITIONS

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ABSTRACT. We establish a necessary and sufficient condition for a heptagonal knot to be figure-8 knot. The condition is described by a set of Radon partitions formed by vertices of the heptagon. In addition we relate this result to the number of nontrivial heptagonal knots in linear embeddings of the complete graph K_7 into \mathbb{R}^3 .

1. INTRODUCTION

An m -component *link* is a union of m disjoint circles embedded in \mathbb{R}^3 . Especially a link with only one component is called a *knot*. Two knots K and K' are said to be *ambient isotopic*, denoted by $K \sim K'$, if there exists a continuous map $h : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$ such that the restriction of h to each $t \in [0, 1]$, $h_t : \mathbb{R}^3 \times \{t\} \rightarrow \mathbb{R}^3$, is a homeomorphism, h_0 is the identity map and $h_1(K_1) = K_2$, to say roughly, K_1 can be deformed to K_2 without intersecting its strand. The ambient isotopy class of a knot K is called the *knot type* of K . Especially if K is ambient isotopic to another knot contained in a plane of \mathbb{R}^3 , then we say that K is *trivial*. The ambient isotopy class of links is defined in the same way.

In this paper we will focus on polygonal knots. A *polygonal knot* is a knot consisting of finitely many line segments, called *edges*. The end points of each edge are called *vertices*. Figure 1 shows polygonal presentations of two knot types 3_1 and 4_1 (These notations for knot types follow the knot tabulation in [16]. Usually 3_1 and its mirror image are called *trefoil*, and 4_1 *figure-8*). For a knot type \mathfrak{K} , its *polygon index* $p(\mathfrak{K})$ is defined to be the minimal number of edges required to realize \mathfrak{K} as a polygonal knot. Generally it is not easy to determine $p(\mathfrak{K})$ for an arbitrary knot type \mathfrak{K} . This quantity was determined only for some specific knot types [3, 7, 9, 12, 15]. Here we mention a result by Randell on small knots.

Theorem 1. [15] $p(\text{trivial knot}) = 3$, $p(\text{trefoil}) = 6$ and $p(\text{figure-8}) = 7$. Furthermore, $p(\mathfrak{K}) \geq 8$ for any other knot type \mathfrak{K} .

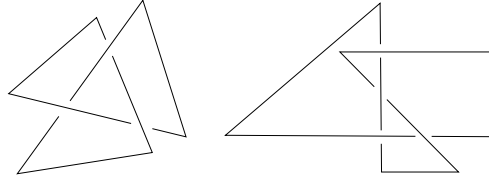


FIGURE 1. Polygonal presentations of 3_1 and 4_1 knots

Let V be a set of points in \mathbb{R}^3 . A partition $V_1 \cup V_2$ of V is called a *Radon partition* if the two convex hulls of V_1 and V_2 intersect each other. For example, if

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V consists of 5 points in general position, then it should have a Radon partition such that $(|V_1|, |V_2|) = (1, 4)$ or $(2, 3)$.

We remark that the notion of Radon partition can be utilized to describe the knot type of a polygonal knot. In [1] a set of Radon partitions are derived from vertices of heptagonal trefoil knots and also hexagonal trefoil knots. Similar work was also done for hexagonal trefoil knot in [8]. These results were effectively applied to investigate knots in linear embeddings of the complete graph K_7 and K_6 . An embedding of a graph into \mathbb{R}^3 is said to be *linear*, if each edge of the graph is mapped to a line segment. In [1] Alfonsín showed that every linear embedding of K_7 contains a heptagonal trefoil knot as its cycle. And in [8] it was proved that the number of nontrivial knots in any linear embedding of K_6 is at most one.

In this paper we give a necessary and sufficient condition for a heptagonal knot to be figure-8 via notion of Radon partition. And we discuss how our result can be utilized to determine the maximal number of heptagonal knots with polygon index 7 residing in linear embeddings of K_7 .

Now we introduce some notations necessary to describe the main theorem. Let P be a heptagonal knot such that its vertices are in general position. We can label the vertices of P by $\{1, 2, \dots, 7\}$ so that each vertex i is connected to $i + 1 \pmod{7}$ by an edge of P , that is, a labeling of vertices is determined by a choice of base vertex and an orientation of P . Given such a labelling of vertices let $\Delta_{i_1 i_2 i_3}$ denote the triangle formed by three vertices $\{i_1, i_2, i_3\}$, and e_{jk} the line segment from the vertex j to vertex k . The relative position of such a triangle and a line segment will be represented via “ ϵ ” which is defined below:

- (i) If $\Delta_{i_1 i_2 i_3} \cap e_{jk} = \emptyset$, then set $\epsilon(i_1 i_2 i_3, jk) = 0$.
- (ii) Otherwise,
 $\epsilon(i_1 i_2 i_3, jk) = 1$ (resp. -1), when $(\overrightarrow{i_1 i_2} \times \overrightarrow{i_2 i_3}) \cdot \overrightarrow{jk} > 0$ (resp. < 0).

The tables in Theorem 2 show the values of ϵ between triangles formed by three consecutive vertices and edges of P . If ϵ is zero, then the corresponding cell in the table is filled by “ \times ”. Otherwise, we mark by “ $+$ ” or “ $-$ ” according to the sign of ϵ . For example, according to *RS-I*, $\epsilon(123, 67) = 0$ and

$$(\epsilon(123, 45), \epsilon(123, 56), \epsilon(234, 56)) = (1, -1, -1) \quad \text{or} \quad (-1, 1, 1).$$

In later sections, for our convenience, we use “ \bullet ” to indicate $\epsilon \neq 0$, without specifying the sign.

Theorem 2. *Let P be a heptagonal knot such that its vertices are in general position. Then P is figure-8 if and only if the vertices of P can be labelled so that the polygon satisfies one among the three types *RS-I*, *RS-II* and *RS-III*.*

123	45	56	67
	\pm	\mp	\times
234	56	67	71
	\mp	\times	\times
345	67	71	12
	\times	\pm	\times
456	71	12	23
	\pm	\times	\times
567	12	23	34
	\times	\mp	\times
671	23	34	45
	\mp	\times	\times
712	34	45	56
	\times	\pm	\times

RS-I

123	45	56	67
	\pm	\mp	\times
234	56	67	71
	\mp	\times	\times
345	67	71	12
	\times	\pm	\times
456	71	12	23
	\pm	\times	\times
567	12	23	34
	\times	\mp	\times
671	23	34	45
	\mp	\pm	\times
712	34	45	56
	\times	\pm	\times

RS-II

123	45	56	67
	\pm	\mp	\times
234	56	67	71
	\times	\mp	\times
345	67	71	12
	\times	\pm	\times
456	71	12	23
	\pm	\times	\times
567	12	23	34
	\times	\mp	\times
671	23	34	45
	\mp	\times	\times
712	34	45	56
	\times	\pm	\times

RS-III

In Section 2 we discuss a possible application of Theorem 2. And the remaining sections will be devoted to the proof of the theorem.

2. HEPTAGONAL KNOTS IN K_7

In 1983 Conway and Gordon proved that every embedding of K_7 into \mathbb{R}^3 contains a nontrivial knot as its cycle [4]. This result was generalized by Negami. He showed that given a knot type \mathfrak{K} there exists a number $r(\mathfrak{K})$ such that every linear embedding of K_n with $n \geq r(\mathfrak{K})$ contains a polygonal knot of type \mathfrak{K} [13].

It would be not easy to determine $r(\mathfrak{K})$ for an arbitrary knot type \mathfrak{K} . But if the knot type is of small polygon index, we may attempt to do. For example, Alfonsín showed that $r(\text{trefoil}) = 7$ [1]. To determine the number, he utilized the theory of oriented matroid. This theory provides a way to describe geometric configurations (See [2]). Any linear embedding of K_7 is determined by fixing the position of seven vertices in \mathbb{R}^3 . The relative positions of these seven points can be described by an uniform acyclic oriented matroid of rank 4 on seven elements which is in fact a collection of Radon partitions, called *signed circuits*, formed by the seven points. Alfonsín constructed several conditions at least one among which should be satisfied if a set of seven points constitutes a heptagonal trefoil knot. These conditions are described by a collection of Radon partitions. And then, by help of a computer program, he verified that each of these matroids satisfies at least one of the conditions. Note that all uniform acyclic oriented matroid of rank 4 on seven elements can be completely listed [5, 6].

On the other hand we may consider another quantity. Let \mathcal{F}_n be the collection of all linear embeddings of the complete graph K_n , and let $c(f)$ be the number of knots with polygon index n in a linear embedding $f \in \mathcal{F}_n$. Define $M(n)$ and $m(n)$ to be

$$M(n) = \text{Max} \{c(f) | f \in \mathcal{F}_n\}, \quad m(n) = \text{Min} \{c(f) | f \in \mathcal{F}_n\}.$$

For $n < 6$ these numbers are meaningless because there is no nontrivial knot whose polygonal index is less than 6. In [8] it was shown that $M(6) = 1$ and $m(n) = 0$ for every n . To determine $M(6)$ the author derived a set of Radon partitions from hexagonal trefoil knot. Since 3_1 and its mirror image are only knot types of polygon index 6, by verifying that the conditional set arises from at most one cycle in any embedded K_6 , the number $M(6)$ was determined.

We remark that also the number $M(7)$ can be determined by applying our main theorem to a procedure as done by Alfonsín. Given a uniform acyclic oriented matroid of rank 4 on seven elements, count the number of permutations which produce any of the conditional partition sets in Theorem 2. Since the figure-8 is the only knot type of polygon index 7 and the condition in the theorem is necessary and sufficient for a heptagonal knot to be figure-8, the counted number is the number of knots with polygon index 7 in a corresponding embedding of K_7 . Hence, by getting the maximum among all such numbers over all uniform acyclic oriented matroids of rank 4 on seven elements, $M(7)$ can be determined.

3. CONWAY POLYNOMIAL

In this section we give a brief introduction on Conway polynomial which is an ambient isotopy invariant of knots and links. This invariant will be utilized to prove the main theorem in later sections. See [11, 10] for more detailed or kind introduction.

Let L be a link. Given a plane N in \mathbb{R}^3 , let $\pi_N : N \times \mathbb{R} \rightarrow N$ be the map defined by $\pi(x, y, z) = (x, y)$. Then π_N is called a *regular projection* of L , if the restricted map $\pi_N : L \rightarrow N$ has only finitely many multiple points and every multiple point is a transversal double point. By specifying which strand goes over

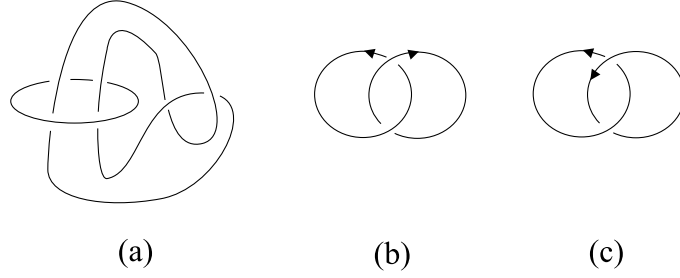


FIGURE 2.

at each double point of the regular projection, we obtain a *diagram* representing L . The double points in a diagram are called *crossings*. Figure 2-(a) shows an example of unoriented link diagram. The diagrams in (b) and (c) represent oriented links. Also the figures in Figure 1 can be considered to be unoriented knot diagrams.

Let $L = L_1 \cup L_2$ be a 2-component oriented link. For each L_i we can choose an oriented surface F_i such that $\partial F_i = L_i$. This surface F_i is called a *Seifert surface* of L_i . Then the linking number $lk(L_1, L_2)$ is defined to be the algebraic intersection number of L_2 through F_1 . It is known that the linking number is independent of the choice of Seifert surface, and $lk(L_1, L_2) = lk(L_2, L_1)$. Hence we may denote the number by $lk(L)$ instead of $lk(L_1, L_2)$. The linking numbers of the links in Figure 2-(b) and (c) are 1 and -1 respectively. The link in (a) is of linking number 0 for any choice of orientation.

Let \mathcal{D} be the collection of diagrams of all oriented links. Then a function $\nabla : \mathcal{D} \rightarrow \mathbb{Z}[t]$ is uniquely determined by the following three axioms:

- (i) Let D and D' be diagrams which represent two oriented links L and L' respectively. If L is ambient isotopic to L' with orientation preserved, then $\nabla(D) = \nabla(D')$.
- (ii) If D is a diagram representing the trivial knot, then $\nabla(D) = 1$.
- (iii) Let D_+ , D_- and D_0 be three diagrams which are exactly same except at a neighborhood of one crossing point. In the neighborhood they differ as shown in Figure 3. The crossing of D_+ (*resp.* D_-) in the figure is said to be *positive* (*resp.* *negative*). Then the following equality, called the *skein relation*, holds:

$$\nabla(D_+) - \nabla(D_-) = t\nabla(D_0)$$

If D is a diagram of an oriented link L , then the *Conway polynomial* $\nabla(L)$ of L is defined to be $\nabla(D)$. Now we give some facts on Conway polynomial which are necessary for our use in later sections.

Lemma 3. [11, 10]

- (i) Let $-K$ be the oriented knot obtained from an oriented knot K by reversing its orientation. Then $\nabla(-K) = \nabla(K)$.
- (ii) If K is trefoil, then $\nabla(K) = 1 + t^2$. And if K is figure-8, then $\nabla(K) = 1 - t^2$.
- (iii) Let L be an oriented link with two components. Then its Conway polynomial is of the form $\nabla(L) = a_1 t + a_2 t^2 + \cdots$ with $a_1 = lk(L)$.

4. RADON PARTITIONS IN HEPTAGONAL FIGURE-8 KNOT

In this section we give several lemmas necessary for the proof of Theorem 2. Throughout this section P is a heptagonal figure-8 knot such that its vertices are in general position and labelled by $\{1, 2, \dots, 7\}$ along an orientation. Some lemmas

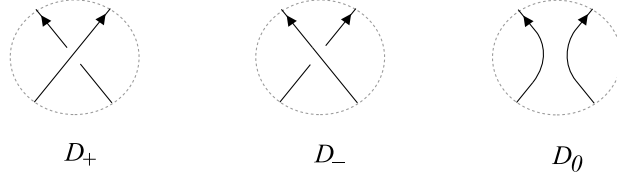


FIGURE 3.

will be described by using tables as in Theorem 2. Note that the blanks in the tables of the following lemma and the rest of this article indicate that the values of ϵ are not decided yet.

Lemma 4. *The following implications hold for P .*

$$\begin{array}{ll}
 (i) \quad \begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & \pm & \times & \times \\ \hline \end{array} & \implies \begin{array}{|c|c|c|c|} \hline 234 & 56 & 67 & 71 \\ \hline & \times & & \\ \hline 345 & 67 & 71 & 12 \\ \hline & \mp & & \\ \hline \end{array} \\
 (ii) \quad \begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & \times & \times & \pm \\ \hline \end{array} & \implies \begin{array}{|c|c|c|c|} \hline 671 & 23 & 34 & 45 \\ \hline & & & \mp \\ \hline 712 & 34 & 45 & 56 \\ \hline & & & \times \\ \hline \end{array}
 \end{array}$$

Proof. Note that (i) is identical with (ii) after relabelling vertices of P along the reverse orientation. Hence it suffices to prove only (i). Assume $\epsilon(123, 45) = 1$. Then we can choose a diagram of P in which e_{23} and e_{45} produce a positive crossing. Figure 4-(a) depicts the diagram partially. Set $K_+ = P$ and apply the skein relation of Conway polynomial so that

$$\nabla(K_+) - \nabla(K_-) = t \nabla(K_0) ,$$

where K_- is the cycle $\langle 12\#34567 \rangle$ and $K_0 = \langle 12\#567 \rangle \cup \langle 34* \rangle$ as seen in Figure 4-(b) and (c). The conditional part of (i) implies that e_{45} is the only edge of P piercing Δ_{123} . Hence $K_- \sim \langle 134567 \rangle$ by an isotopy in Δ_{123} . Similarly $K_0 \sim \langle 1\#567 \rangle \cup \langle 34* \rangle$. Since $\langle 134567 \rangle$ is a hexagon, K_- should be trivial or trefoil by Theorem 1. Therefore, $\nabla(K_-) = 1$ or $1 + t^2$ and because $\nabla(K_+) = 1 - t^2$, we have

$$\nabla(K_0) = -t \quad \text{or} \quad -2t .$$

By Lemma 3-(iii) at least one edge of $\langle 1\#567 \rangle$ penetrates Δ_{34*} in negative direction. Note that Δ_{34*} is contained in a half space H_{123}^- with respect to the plane H_{123}^0 formed by $\{1, 2, 3\}$. Since $e_{1\#}$ belongs to Δ_{123} and $e_{\#5}$ belongs to another half space H_{123}^+ , the two edges are excluded from candidates. Also e_{56} is excluded because $\Delta_{34*} \subset \Delta_{345}$. Hence e_{67} and e_{71} are the only edges which may penetrate Δ_{34*} . But the vertex 1 belongs to H_{34*}^+ , which implies that if e_{71} penetrates Δ_{34*} , then the orientation of intersection should be positive. Therefore we can conclude $\epsilon(34*, 67) = -1$, and hence $\epsilon(345, 67) = -1$.

Let $T_{5,123}^\infty$ be the set of all half infinite lines starting the vertex 5 and passing through a point of Δ_{123} . Clearly $\Delta_{234} \subset T_{5,123}^\infty$. Hence if we suppose $\epsilon(234, 56) \neq 0$, then also $\epsilon(123, 56) \neq 0$, which is contradictory to the condition of (i).

In the case that $\epsilon(123, 45) = -1$ we can prove the implication in a similar way. \square

Lemma 5.

$$\begin{array}{ll}
 (i) \quad \epsilon(123, 56) = \pm 1 \quad \text{and} \quad \epsilon(456, 12) \neq 0 & \implies \quad \epsilon(456, 12) = \pm 1 \\
 (ii) \quad \epsilon(123, 56) = \pm 1 \quad \text{and} \quad \epsilon(567, 23) \neq 0 & \implies \quad \epsilon(567, 23) = \pm 1
 \end{array}$$

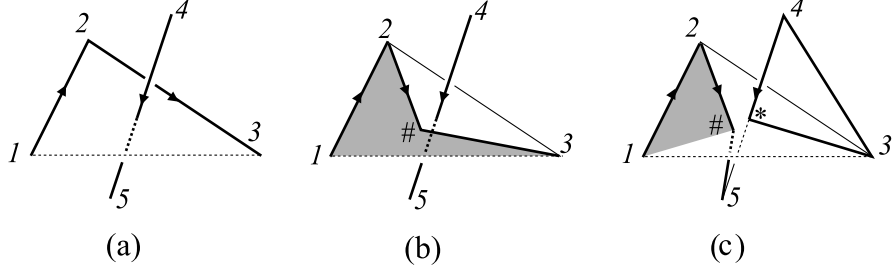
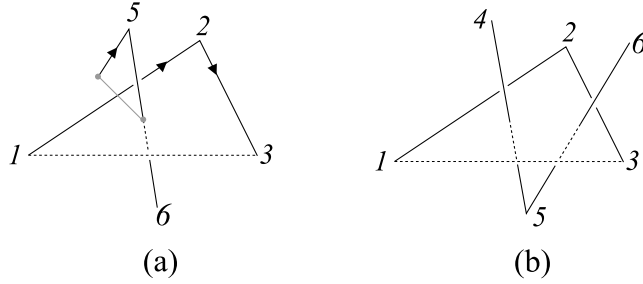
FIGURE 4. K_+ , K_- and K_0 

FIGURE 5.

Proof. Assuming $\epsilon(123, 56) = 1$, the conditional part of (i) can be illustrated as Figure 5-(a). From the figure we can verify (i). Similarly (ii) can be proved. \square

Lemma 6.

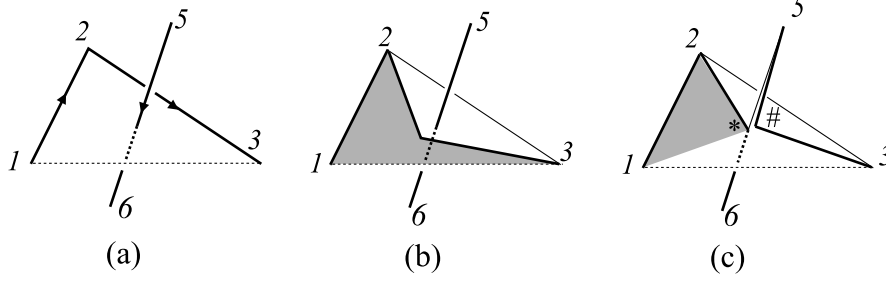
$$\begin{aligned}
 (i) \quad & \begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & \bullet & \bullet & \\ \hline \end{array} \implies \begin{array}{|c|c|c|c|} \hline 345 & 67 & 71 & 12 \\ \hline & & & \times \\ \hline 456 & 71 & 12 & 23 \\ \hline & \bullet & \times & \times \\ \hline \end{array} \\
 (ii) \quad & \begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & & \bullet & \bullet \\ \hline \end{array} \implies \begin{array}{|c|c|c|c|} \hline 567 & 12 & 23 & 34 \\ \hline & \times & \times & \bullet \\ \hline 671 & 23 & 34 & 45 \\ \hline & \times & & \\ \hline \end{array}
 \end{aligned}$$

Proof. Assuming $\epsilon(123, 45) = 1$, the conditional part of (i) can be illustrated as Figure 5-(b). The figure clearly shows that $\epsilon(345, 12) = 0$, $\epsilon(456, 12) = 0$ and $\epsilon(456, 23) = 0$. Note that e_{71} , e_{12} and e_{23} are the only possible edges of P which may penetrate Δ_{456} . Hence $\epsilon(456, 71)$ should be nonzero. Otherwise, $P = \langle 1234567 \rangle$ can be isotoped to the hexagon $\langle 123467 \rangle$ along Δ_{456} , which contradicts that P is of polygon index 7 by Theorem 1. Similarly (ii) can be proved. \square

Lemma 7. P does not allow any of two cases below:

$$\begin{aligned}
 (i) \quad & \begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & \times & \pm & \times \\ \hline 345 & 67 & 71 & 12 \\ \hline & \times & \pm & \\ \hline \end{array} \quad (ii) \quad \begin{array}{|c|c|c|c|} \hline 671 & 23 & 34 & 45 \\ \hline & & \pm & \times \\ \hline 123 & 45 & 56 & 67 \\ \hline & \times & \pm & \times \\ \hline \end{array}
 \end{aligned}$$

Proof. Suppose that (i) is true. It suffices to consider the case that $\epsilon(123, 56) = \epsilon(345, 71) = 1$. Apply the skein relation to the crossing between e_{23} and e_{56} as seen in Figure 6, so that $K_- \sim \langle 134567 \rangle$ and $K_0 \sim \langle 1 * 67 \rangle \cup \langle 5 \# 34 \rangle$. Then $\nabla(K_0)$

FIGURE 6. K_+ , K_- and K_0

should be $-t$ or $-2t$, that is, the linking number of K_0 is -1 or -2 . Note that $\Delta_{5\#3} \cup \Delta_{345}$ is a Seifert surface of $\langle 5\#34 \rangle$. Therefore

$$\sum_{i \in \{1*, *6, 67, 71\}} (\epsilon(5\#3, e_i) + \epsilon(345, e_i)) = -1 \text{ or } -2.$$

By our assumption $\epsilon(345, 67) = 0$ and $\epsilon(345, 71) = 1$. Clearly we know that $\epsilon(5\#3, e_i) = 0$ for $i = 1*, *6$. Also $\epsilon(345, *6)$ is 0 because e_{*6} is a segment of e_{56} . Select the point $\#$ so that $\Delta_{5\#3} \subset \Delta_{356}$, hence $\epsilon(5\#3, 67)$ is 0. Therefore the summation should be -1 , and $\epsilon(345, 1*) = \epsilon(5\#3, 71) = -1$. But the vertex 1 belongs to $H_{5\#3}^+$ as seen in the figure, hence if e_{71} penetrates $\Delta_{5\#3}$, then the orientation of intersection should be positive, which is a contradiction.

(ii) is derived directly from (i) by relabelling the vertices after reversing the orientation of P . \square

Two integers i and j indicate the same vertex if $i \equiv j \pmod{7}$. For an integer i , define $I(i)$ to be the number of edges of P penetrating $\Delta_{i, i+1, i+2}$, that is,

$$I(i) = \sum_{j \in \{i+3, i+4, i+5\}} |\epsilon((i, i+1, i+2), (j, j+1))|.$$

Lemma 8. *There exists no integer i such that $I(i) \geq 2$ and $I(i+1) \geq 2$.*

Proof. Suppose that $I(1) \geq 2$ and $I(2) \geq 2$. Then, up to the relabelling $(1, 2, 3, 4, 5, 6, 7) \rightarrow (4, 3, 2, 1, 7, 6, 5)$

it is enough to observe the following six cases:

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For Case (i), apply Lemma 6 to Δ_{123} and Δ_{234} . Then we have

456	71	12	23
	•	×	×
567	12	23	34
	•	×	×

But, applying Lemma 4-(i) to Δ_{456} , $\epsilon(567, 12)$ should be 0, a contradiction. Also Case (iii) can be rejected in a similar way. For Case (vi) to be excluded, only Lemma 6 is enough.

For Case (ii) we may assume further that $\epsilon(123, 45) = 1$. Then, for e_{71} to penetrate Δ_{234} , the vertex 7 should belong to H_{123}^- . This implies that the region

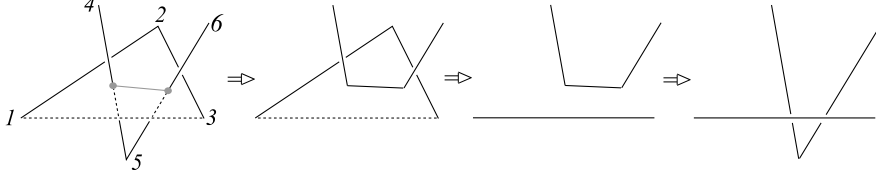


FIGURE 7.

$\Delta_{456} \cap H_{123}^+$ is not penetrated by any edge of P , and e_{45} and e_{56} are the only edges of P penetrating Δ_{123} . Hence we can isotope P to $\langle 134567 \rangle$ as illustrated in Figure 7, which contradicts Theorem 1. Also in Case (iv) P can be isotoped to $\langle 245671 \rangle$ in a similar way.

For case (v), we may suppose further that $\epsilon(123, 45) = 1$. Then Δ_{234} belongs to H_{123}^- . If $\epsilon(123, 67) = 1$, then e_{71} belongs to H_{123}^+ and hence can not penetrate Δ_{234} , a contradiction. Similarly if $\epsilon(123, 67) = -1$, then e_{56} can not penetrate Δ_{234} . \square

Lemma 9. *There exists no pair of distinct integers (i, j) such that*

$i, i+1, i+2$	$i+3, i+4$	$i+4, i+5$	$i+5, i+6$
	•	•	×
$j, j+1, j+2$	$j+3, j+4$	$j+4, j+5$	$j+5, j+6$
	×	•	•

Proof. By Lemma 8 it is enough to observe the four cases: $(i, j) = (1, 3), (1, 4), (1, 5)$ and $(1, 6)$. The first two cases are contradictory to Lemma 6. For the fourth case, applying Lemma 6 to Δ_{123} , we have

456	71	12	23
	•	×	×

And apply Lemma 4 to Δ_{456} , to have $\epsilon(671, 23) \neq 0$, a contradiction.

Lastly suppose $(i, j) = (1, 5)$. Then, we can observe which edges penetrate Δ_{671} and Δ_{712} as follows:

123	45	56	67	Lemma 6	234	56	67	71	Lemma 4	671	23	34	45
	•	•	×			×	×	•			•		
567	12	23	34	\Rightarrow	456	71	12	23	\Rightarrow	712	34	45	56
	×	•	•			•	×	×					•
Lemma 8				\Rightarrow					$(i, j) = (1, 5)$				
					671	23	34	45					
						•	×	×		712	34	45	56
						×	×	•			×	×	•

We may assume $\epsilon(123, 45) = 1$. Then clearly $\epsilon(123, 56) = -1$, and by Lemma 4 $\epsilon(671, 23) = -1$. Now we apply the skein relation to $e_{23} \cup e_{67}$ as seen in Figure 8 so that $K_- = P$, $K_0 \sim \langle 6\#345 \rangle \cup \langle 2 * 1 \rangle$ and $\Delta_{2*1} \subset \Delta_{123}$. Then immediately it is observed that

$$\epsilon(2 * 1, 6\#) = \epsilon(2 * 1, \#3) = \epsilon(2 * 1, 34) = 0$$

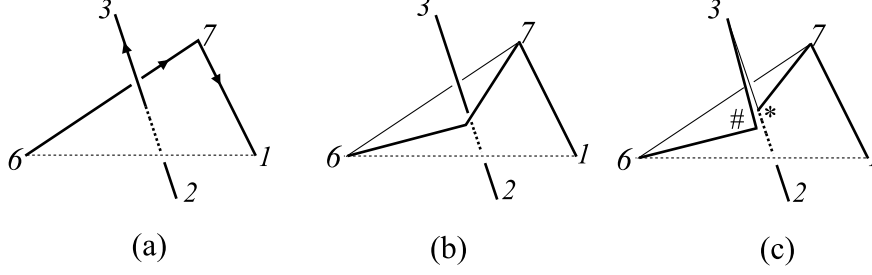
Recall $\epsilon(123, 56) = -1$, which implies that if $\epsilon(2 * 1, 56) \neq 0$, then the value should be negative. Therefore, considering $\nabla(K_0) = t$ or $2t$, it should hold that

$$\epsilon(2 * 1, 45) = 1, \quad \epsilon(2 * 1, 56) = 0$$

and e_{56} penetrates $\Delta_{*31} = \Delta_{123} - \Delta_{2*1}$. Since $\Delta_{*31} \subset H_{671}^-$, the vertex 5 belongs to H_{671}^- . Hence $e_{56} \subset H_{671}^-$, which is contradictory to $\epsilon(712, 56) \neq 0$ because Δ_{712} belongs to the other half space H_{671}^+ . \square

Lemma 10.

(i) For every i , $I(i) \geq 1$.

FIGURE 8. K_- , K_+ and K_0

(ii) There exists an integer i such that $I(i) \geq 2$.

(iii) For every i , $I(i) < 3$.

(iv) If $I(i) = 2$ for some i , then $e_{i+4, i+5}$ should penetrate $\Delta_{i, i+1, i+2}$.

Proof of (i). Suppose $I(1) = 0$, that is, Δ_{123} is not penetrated by any edge of P . Then we can isotope P along Δ_{123} , so that $P \sim \langle 134567 \rangle$. By Theorem 1, the hexagon $\langle 134567 \rangle$ is trivial or trefoil, a contradiction. \square

Proof of (ii). Suppose $I(i) = 1$ for every i . Then, among e_{45} , e_{56} and e_{67} , only one edge penetrates Δ_{123} . Firstly assume that e_{45} does. Then, applying Lemma 4 repeatedly, we have a sequence of implications:

$$\begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & \bullet & \times & \times \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|} \hline 345 & 67 & 71 & 12 \\ \hline & \bullet & \times & \times \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|} \hline 567 & 12 & 23 & 34 \\ \hline & \bullet & \times & \times \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|} \hline 712 & 34 & 45 & 56 \\ \hline & \bullet & \times & \times \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|} \hline 234 & 56 & 67 & 71 \\ \hline & \bullet & \times & \times \\ \hline \end{array}$$

But, by Lemma 4 again, the first and last tables are contradictory to each other. The case that e_{67} penetrates the triangle is rejected in a similar way.

Now it can be assumed that every $\Delta_{i, i+1, i+2}$ is penetrated only by $e_{i+4, i+5}$. Then, applying Lemma 5 repeatedly, we have that

$$\begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & \times & \pm & \times \\ \hline \end{array} \text{ and } \begin{array}{|c|c|c|c|} \hline 345 & 67 & 71 & 12 \\ \hline & \times & \pm & \times \\ \hline \end{array},$$

which is contradictory to Lemma 7. \square

Proof of (iii). Suppose $I(1) = 3$. Then we have two implications as follows:

$$\begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & \bullet & \bullet & \\ \hline \end{array} \xRightarrow{\text{Lemma 6}} \begin{array}{|c|c|c|c|} \hline 456 & 71 & 12 & 23 \\ \hline & \bullet & \times & \times \\ \hline \end{array} \xRightarrow{\text{Lemma 4}} \begin{array}{|c|c|c|c|} \hline 671 & 23 & 34 & 45 \\ \hline & \bullet & & \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & & \bullet & \bullet \\ \hline \end{array} \xRightarrow{\text{Lemma 6}} \begin{array}{|c|c|c|c|} \hline 671 & 23 & 34 & 45 \\ \hline & \times & & \\ \hline \end{array}$$

But these are contradictory to each other. \square

Proof of (iv). Suppose that Δ_{123} satisfies the following:

$$\begin{array}{|c|c|c|c|} \hline 123 & 45 & 56 & 67 \\ \hline & \bullet & \times & \bullet \\ \hline \end{array}$$

Then it is enough to observe two cases $(\epsilon(123, 45), \epsilon(123, 67)) = (1, -1)$ and $(1, 1)$. These cases are depicted as in Figure 9-(a) and (b) respectively. In the first case 5 and 6 are the only vertices which belong to H_{123}^+ . Therefore P can be isotoped to $\langle 1234 * \# 7 \rangle$ along the tetragon formed by $\{*, 5, 6, \#\}$. And lift $e_{*\#}$ slightly into H_{123}^- , then we have $I(1) = 0$ for the resulting heptagon, a contradiction.

For the second case we observe which edges penetrate Δ_{234} . Note that $\Delta_{234} \subset T_{5, 123}^\infty$. Hence if a line starting at the vertex 5 penetrates Δ_{234} , then it also penetrates Δ_{123} . This implies $\epsilon(234, 56) = 0$. Also $\epsilon(234, 71) = 0$, because e_{71} belongs to H_{123}^+ but Δ_{234} belongs to the other half space H_{123}^- . Therefore e_{67} is the only

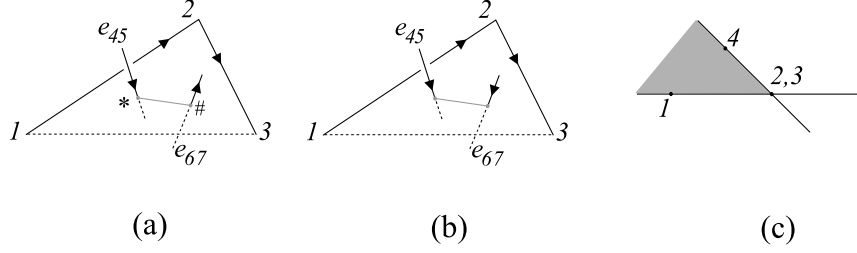


FIGURE 9.

edge of P penetrating Δ_{234} . Furthermore the orientation of intersection should be positive. This can be seen easily from Figure 9-(c). Let N be a plane in \mathbb{R}^3 orthogonal to $\vec{23}$. And let $\pi : \mathbb{R}^3 \cong N \times \mathbb{R} \rightarrow N$ be the orthogonal projection onto N such that the vertex 3 is above the vertex 2 with respect to the \mathbb{R} -coordinate. Figure 9-(c) depicts the image of $H_{123}^0 \cup H_{234}^0$ under π . Suppose $\epsilon(234, 67) = -1$. Since $\epsilon(123, 67) = 1$, the vertex 6 should belong to $H_{123}^- \cap H_{234}^+$ which corresponds to the shaded region in the figure. Then, as seen in the figure, it is impossible that e_{67} penetrates both Δ_{123} and Δ_{234} .

In a similar way e_{45} should be the only edge of P penetrating Δ_{712} and the orientation of intersection is positive. To summarize, we have

234	56	67	71
	×	+	×

and

712	34	45	56
	×	+	×

This contradicts Lemma 7. □

5. PROOF OF THEOREM 2

We prove the “only if” part of Theorem 2 by filling in the table of penetrations in P . By Lemma 10 it can be assumed that

123	45	56	67
	•	•	×

Applying Lemma 6 to Δ_{123} and Lemma 4 to Δ_{456} , we have the initial status S_0 as shown in Table 1. Considering Lemma 9, we know that the row of 567 should be filled by $(\times, \bullet, \times)$ or $(\times, \times, \bullet)$. The second is excluded by Lemma 4. Lemma 10-(iv) guarantees $\epsilon(671, 45) = 0$. Hence the status S'_0 is derived. Observe how the row of 712 can be filled. $I(7)$ should be 1 or 2 by Lemma 10-(i) and (iii). In fact $I(7)$ should be 1 by Lemma 8. And $(\bullet, \times, \times)$ and $(\times, \times, \bullet)$ are disallowed by Lemma 4, hence S''_0 is derived.

In a similar way we know that 234 should have $(\bullet, \times, \times)$ or $(\times, \bullet, \times)$. Hence S''_0 can proceed to the status S_1 or S_2 .

Case 1: the row of 234 is filled with $(\bullet, \times, \times)$. Observe 345 in S_1 . Apply Lemma 4 to Δ_{234} . Then $\epsilon(345, 67) = 0$, and $\epsilon(345, 71) \neq 0$. Therefore we obtain S'_1 . Finally, let S'_{1-1} (resp. S'_{1-2}) be the status obtained from S'_1 by setting $\epsilon(671, 34)$ to be zero (resp. nonzero).

Note that if the table is completely filled with “•” and “×”, then the orientation of intersection is automatically determined. See S'_{1-1} . Since 123 has $(\bullet, \bullet, \times)$, the possible orientation is $(+, -, \times)$ or $(-, +, \times)$. Assume the former. Applying Lemma 4 to Δ_{671} , we know $\epsilon(671, 23) = -1$. Also applying the lemma to Δ_{456} and Δ_{234} , we have $\epsilon(456, 71) = 1$ and $\epsilon(234, 56) = -1$. Furthermore, from the assumption $\epsilon(123, 56) = -1$, it is derived that $\epsilon(567, 23) = -1$ and $\epsilon(345, 71) = 1$ by Lemmas 5 and 7 respectively. Similarly $\epsilon(712, 45) = 1$. Therefore S'_{1-1} is identical with $RS-I$.

For S'_{1-2} , first determine the orientations in the second column following the method used above. Then, under the assumption $\epsilon(123, 56) = -1$, it should hold

<table><tr><td>123</td><td>45</td><td>56</td><td>67</td></tr><tr><td></td><td>•</td><td>•</td><td>×</td></tr><tr><td>234</td><td>56</td><td>67</td><td>71</td></tr><tr><td></td><td>•</td><td>×</td><td>×</td></tr><tr><td>345</td><td>67</td><td>71</td><td>12</td></tr><tr><td></td><td>71</td><td>12</td><td>23</td></tr><tr><td>456</td><td>•</td><td>×</td><td>×</td></tr><tr><td></td><td>12</td><td>23</td><td>34</td></tr><tr><td>567</td><td>×</td><td>•</td><td>×</td></tr><tr><td></td><td>23</td><td>34</td><td>45</td></tr><tr><td>671</td><td>•</td><td>×</td><td>×</td></tr><tr><td></td><td>34</td><td>45</td><td>56</td></tr><tr><td>712</td><td>×</td><td>•</td><td>×</td></tr></table> S_0	123	45	56	67		•	•	×	234	56	67	71		•	×	×	345	67	71	12		71	12	23	456	•	×	×		12	23	34	567	×	•	×		23	34	45	671	•	×	×		34	45	56	712	×	•	×	\Rightarrow	<table><tr><td>123</td><td>45</td><td>56</td><td>67</td></tr><tr><td></td><td>•</td><td>•</td><td>×</td></tr><tr><td>234</td><td>56</td><td>67</td><td>71</td></tr><tr><td></td><td>•</td><td>×</td><td>×</td></tr><tr><td>345</td><td>67</td><td>71</td><td>12</td></tr><tr><td></td><td>71</td><td>12</td><td>23</td></tr><tr><td>456</td><td>•</td><td>×</td><td>×</td></tr><tr><td></td><td>12</td><td>23</td><td>34</td></tr><tr><td>567</td><td>×</td><td>•</td><td>×</td></tr><tr><td></td><td>23</td><td>34</td><td>45</td></tr><tr><td>671</td><td>•</td><td>×</td><td>×</td></tr><tr><td></td><td>34</td><td>45</td><td>56</td></tr><tr><td>712</td><td>×</td><td>•</td><td>×</td></tr></table> S'_0	123	45	56	67		•	•	×	234	56	67	71		•	×	×	345	67	71	12		71	12	23	456	•	×	×		12	23	34	567	×	•	×		23	34	45	671	•	×	×		34	45	56	712	×	•	×	\Rightarrow	<table><tr><td>123</td><td>45</td><td>56</td><td>67</td></tr><tr><td></td><td>•</td><td>•</td><td>×</td></tr><tr><td>234</td><td>56</td><td>67</td><td>71</td></tr><tr><td></td><td>•</td><td>×</td><td>×</td></tr><tr><td>345</td><td>67</td><td>71</td><td>12</td></tr><tr><td></td><td>71</td><td>12</td><td>23</td></tr><tr><td>456</td><td>•</td><td>×</td><td>×</td></tr><tr><td></td><td>12</td><td>23</td><td>34</td></tr><tr><td>567</td><td>×</td><td>•</td><td>×</td></tr><tr><td></td><td>23</td><td>34</td><td>45</td></tr><tr><td>671</td><td>•</td><td>×</td><td>×</td></tr><tr><td></td><td>34</td><td>45</td><td>56</td></tr><tr><td>712</td><td>×</td><td>•</td><td>×</td></tr></table> S'_0	123	45	56	67		•	•	×	234	56	67	71		•	×	×	345	67	71	12		71	12	23	456	•	×	×		12	23	34	567	×	•	×		23	34	45	671	•	×	×		34	45	56	712	×	•	×																																																				
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TABLE 1.

that $\epsilon(671, 34) = 1$. This implies $\epsilon(671, 23) = -1$, from which the orientations in the first column can be determined. In this way we can verify that S'_{1-2} is identical with $RS-II$.

Case 2: the row of 234 is filled with $(\times, \bullet, \times)$. If 671 has $(\bullet, \bullet, \times)$ in S_2 , then 234 should have $(\bullet, \times, \times)$ by Lemma 6, a contradiction. Therefore S_2 proceeds only to S'_2 . Suppose that 345 can be filled with $(\bullet, \times, \times)$. Then also we can derive a contradiction by applying Lemma 4 to Δ_{345} . Hence, in S'_2 , 345 should have $(\bullet, \bullet, \times)$ or $(\times, \bullet, \times)$. In the former case we have S'_{2-1} which becomes $RS-II$ after relabelling vertices by the cyclic permutation sending $(3, 4, 5)$ to $(1, 2, 3)$. In the latter we have S'_{2-2} which is $RS-III$.

Now we prove the “if” part of the theorem. Suppose P is a heptagonal knot satisfying $RS-I$, II or III . Let N be a plane orthogonal to $\vec{23}$, and $\pi : \mathbb{R}^3 \equiv N \times \mathbb{R} \rightarrow N$ be the orthogonal projection onto N such that the vertex 3 is above the vertex 2 with respect to the \mathbb{R} -coordinate. We will construct a diagram of P from the projected image $\pi(P)$. Without loss of generality it can be assumed that $\epsilon(123, 45) = 1$ and $\epsilon(123, 56) = -1$. Then, since the vertex 3 is above 2, the edge e_{45} should pass above e_{12} as illustrated in Figure 10.

Suppose P corresponds to $RS-I$. Then similarly e_{56} passes above e_{12} and below e_{34} . Note that if 7 belongs to H_{123}^- , then e_{23} can not penetrate Δ_{671} . Hence $7 \in H_{123}^+$. Since $\epsilon(567, 23) = -1$, the point $\pi(e_{23})$ should belong to $\pi(\Delta_{567})$. Therefore the point $\pi(7)$ belongs to the shaded region shown in Figure 11-(a).

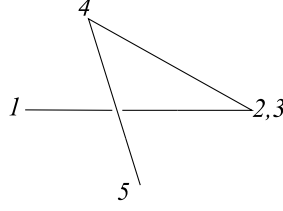


FIGURE 10.

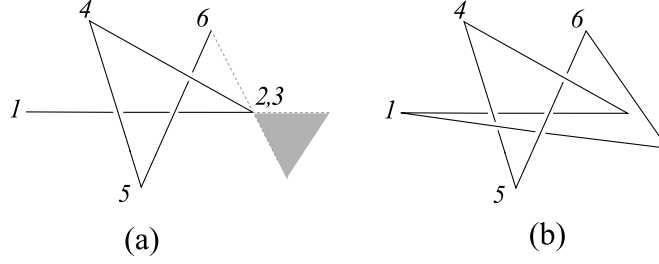


FIGURE 11.

From this we know that $\pi(7) \notin \pi(\Delta_{456})$. And clearly $\pi(1) \notin \pi(\Delta_{456})$. Hence, for $\epsilon(456, 71)$ to be nonzero, $\pi(e_{71})$ should intersect both $\pi(e_{45})$ and $\pi(e_{56})$. In fact e_{71} passes above e_{45} because $\epsilon(712, 45) = 1$ and e_{45} passes above e_{12} . Therefore, since $\epsilon(456, 71)$ is nonzero, e_{71} should pass below e_{56} . The resulting diagram represents figure-8 as seen in Figure 11-(b). Also when P corresponds to $RS-II$, we can obtain a diagram of figure-8 in the same way.

Suppose P corresponds to $RS-III$. Again assume $\epsilon(123, 45) = 1$ and $\epsilon(123, 56) = -1$. Then also in this case we have that $6 \in H_{123}^-$ and $7 \in H_{123}^+$. Especially it should hold that $6 \in H_{234}^+$ and $7 \in H_{234}^-$, because $\epsilon(234, 67) = -1$. Hence the vertex 6 is projected into the shaded region in the top-left of Figure 12-(a) and the vertex 7 into the bottom-right. Now we observe two possible cases according to the position of $\pi(6)$ with respect to $\pi(e_{45})$ as shown in Figure 12-(b) and (c). Again since $\epsilon(234, 67)$ is nonzero, e_{67} should pass below e_{34} in both diagrams. As discussed in the case of $RS-I$, from $\epsilon(456, 71) = 1$, we know that e_{71} passes above e_{45} and below e_{56} . Then the resulting diagram in (b) represents figure-8. In (c), for e_{71} to pass above e_{45} and below e_{56} , the two vertices 6 and 7 should belong to H_{145}^- , which implies that e_{67} passes above e_{45} . Therefore also the resulting diagram in (c) represents figure-8.

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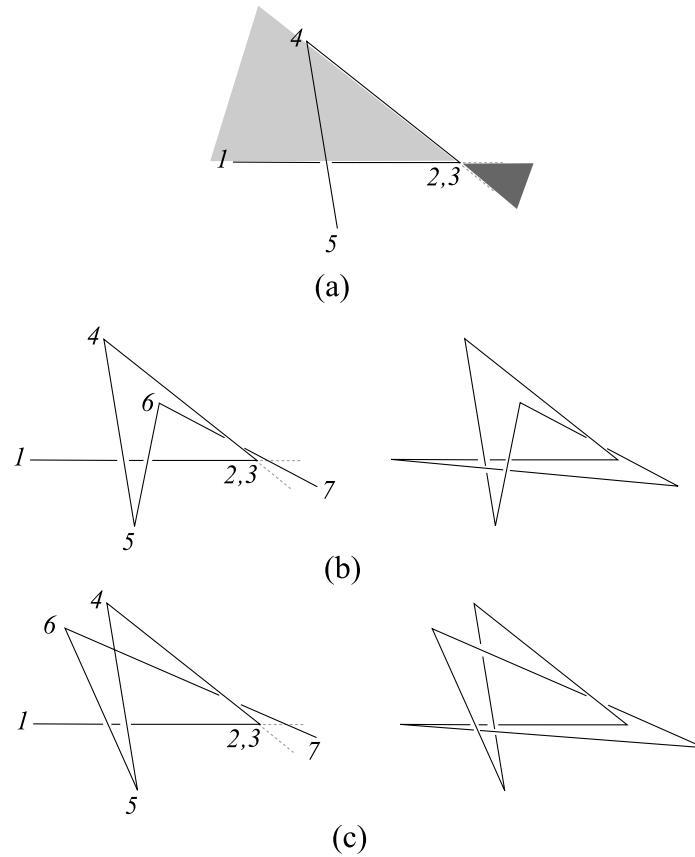


FIGURE 12.

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